Effective integration of Lie type algebras

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Conference in Memory of Yuri Manin

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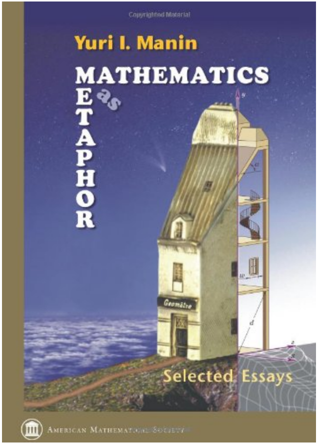
Books

- Yuri I. Manin, *Frobenius manifolds, quantum cohomology, and moduli spaces*, volume 47, American Mathematical Society Colloquium Publications, 1999.
- Sergei I. Gelfand and Yuri I. Manin, *Methods of homological algebra*, Springer Monographs in Mathematics, 2003.
- Alexei I. Kostrikin and Youri I. Manin, *Algèbre et géométrie linéaires*, volume 36, Enseign. Math., Cassini, 2021.
- ...
- Yuri I. Manin, *Mathematics as metaphor: Selected* essays of Yuri. I. Manin, with foreword by Freeman J. Dyson, American Mathematical Society, 2007.
- Yuri I. Manin, *Les mathématiques comme métaphore. Essais choisis*. Les Belles Lettres (Paris), 2021.

Mathematics as metaphor



"The revival of operad theory [...] seems to be a major recent event in the somewhat backwaterish domain of general algebra."

Operads

“GENERAL ALGEBRA”:

- generating operations: binary product $\star$ or skew-symm. bracket $[\,,\,]$.
- relations: associativity $(a \star b) \star c = a \star (b \star c)$ or Jacobi identity $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$.

PROBLEM:

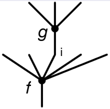
How many different iterations, other structural relations? $\leftrightarrow$ free algebra

- ASSOCIATIVE: $a^{\star n} = a \star a \star \cdots \star a$, $\quad$ $\mathrm{Ass}(x,y) \cong \mathbb{K}\langle\langle x,y\rangle\rangle$.
- LIE: $[[a,b],c] \neq [a,[b,c]]$, ... $\quad$ $\mathrm{Lie}(x,y) \cong \cdots$.

PARADIGM SHIFT: encode the entire set of operations with compositions.

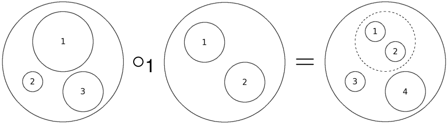
Definition (Operad)

- OPERATIONS: $\{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ of $\mathbb{S}_n$-modules
- COMPOSITIONS: $\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n+m-1)$



Operads

EXAMPLE: Little discs operad D^2



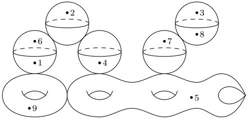
Theorem (May, 1972)

$$Y \sim \Omega^2 X := Top_*(S^2, X) \Longleftrightarrow Y : D^2\text{-}algebra$$

RENAISSANCE OF OPERADS (EARLY 1990'S):

Topology → Algebra, Geometry, Mathematical Physics, etc.

DELIGNE–MUMFORD $\overline{\mathcal{M}}_{g,n}$:



Definition (Kontsevich–Manin, 1994)

$H_\bullet(\overline{\mathcal{M}}_{g,n})$-algebra : Cohomological Field Theories (CohFT)

Renaissance of operads

$\rightarrow$ structure of the Gromov–Witten invariants on $H^\bullet(X)$.

- Maxim Kontsevich and Yuri I. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys., 164(3):525–562, 1994.
- Maxim Kontsevich and Yuri I. Manin, *Quantum cohomology of a product*, with an appendix by Ralf Kaufmann, Invent. Math., 124, 1996.
- Mikhail Kapranov and Yuri I. Manin, *Modules and Morita theorem for operads*, Amer. J. Math., 123(5):811–838, 2001.

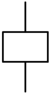
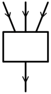
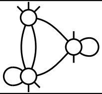
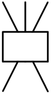
Proposition

$\mathcal{P}$ *operad* $\Rightarrow \left(\prod_{n\in\mathbb{N}} \mathcal{P}(n), \star = \sum_i \circ_i\right)$ *is a pre-Lie algebra:*

$$(a\star b)\star c - a\star(b\star c) = (a\star c)\star b - a\star(c\star b).$$

$$\boxed{\text{associative alg.} \subset \text{pre-Lie alg.} \xrightarrow{\overline{}} \text{Lie alg.}}$$

Generalised operads

type of operations	type of Operads	examples of representations
	associative algebras	Steenrod squares, multicomplexes, ...
	operads	associative alg., Lie alg., pre-Lie alg., Poisson alg., Batalin–Vilkovisky alg., ...
	modular operads	CoFTs, Frobenius algebras, ...
	properads	associative bialg., Frobenius bialg., (involutive) Lie bialg., double Poisson (bi)alg., pre-Calabi–Yau (bi)alg., Airy structures, ...

- Dennis V. Borisov and Yuri I. Manin. *Generalized operads and their inner cohomomorphisms*, in Geometry and dynamics of groups and spaces, volume 265, Progr. Math., p. 247–308. Birkhäuser, 2008.

Quadratic data

THEORY OF "PRESENTATIONS" for (associative, commutative, Lie) algebras: *quadratic data* (V, R) s.t. $R \subset V^{\otimes 2} \Rightarrow A = T(V)/(R)$.
→ category structure, monoidal products (black and white), etc.

- Yuri I. Manin, *Some remarks on Koszul algebras and quantum groups*, Ann. Inst. Fourier (Grenoble), 37(4):191–205, 1987.
- Yuri I. Manin, *Quantum groups and noncommutative geometry*, Université de Montréal Centre de Recherches Mathématiques, 1988.

MIX THE TWO APPROACHES: operadic structure on quadratic data.

- Yuri I. Manin and Bruno Vallette, *Monoidal structures on the categories of quadratic data*, Doc. Math., 25:1727–1786, 2020.

Theorem (Manin–V., 2020)

Drinfeld–Khono quadratic data forms the smallest sub-operad of the Kontsevich graph operad.

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Lie theory

- LIE THIRD THEOREM: Lie group $G \xrightarrow{\mathrm{T}_e}$ Lie algebra $\mathfrak{g}$
- DEFORMATION THEORY: differential graded Lie algebra $(\mathfrak{g}, [\,,\,], \mathrm{d})$

Definition (Maurer–Cartan elements)

$$\mathrm{MC}(\mathfrak{g}) := \left\{\alpha \in \mathfrak{g}_{-1} \mid \mathrm{d}\alpha + \tfrac{1}{2}[\alpha, \alpha] = 0\right\}$$

→ PHILOSOPHY: "any deformation problem over a field of characteristic 0 can be encoded by a dg Lie algebra."

structures of type $\mathcal{P}$ on a "space" A	$\longleftrightarrow$	$\mathrm{MC}(\mathfrak{g}_{\mathcal{P},A})$
equivalence	$\longleftrightarrow$	G

Theorem (Pridham & Lurie, 2010)

equivalence of ∞-categories: formal moduli problems $\overset{\cong}{\longleftrightarrow}$ dg Lie alg.

→ characteristic $p > 0$ (Brantner–Mathew, 2019) , characteristic $p \geqslant 0$ overall operadic proof (Roca i Lucio–Le Grignou, 2023).

Deformation theory

$\to$ complete dg Lie algebra $\mathfrak{g} = \mathcal{F}_1\mathfrak{g} \supset \mathcal{F}_2\mathfrak{g} \supset \cdots$ s.t. $\mathfrak{g} \cong \varprojlim_k \mathfrak{g}/\mathcal{F}_k\mathfrak{g}$.

gauges: $\lambda \in \mathfrak{g}_0 \mapsto$ vector fields: $-\mathrm{d}\lambda + \mathrm{ad}_\lambda \in \Gamma(\mathrm{T}\,\mathrm{MC}(\mathfrak{g}))$

Definition (Gauge equivalence)

$\alpha \sim \beta \in \mathrm{MC}(\mathfrak{g})$: $\exists \lambda \in \mathfrak{g}_0$, $\gamma'(\mathrm{t}) = \mathrm{ad}_\lambda(\gamma(\mathrm{t})) - \mathrm{d}\lambda$, $\gamma(0) = \alpha$, $\gamma(1) = \beta$.

SOLUTION: $\gamma(\mathrm{t}) = \exp(\mathrm{t}\,\mathrm{ad}_\lambda) \;\Rightarrow\; \beta = \exp(\mathrm{ad}_\lambda)(\alpha) + \dfrac{\mathrm{id} - \exp(\mathrm{ad}_\lambda)}{\mathrm{ad}_\lambda}(\mathrm{d}\lambda)$.

SPECIAL CASE: $[a, b] = a \star b - (-1)^{|a||b|} b \star a$, $(\mathfrak{g}, \star, \mathrm{d})$ dg associative alg.

- Maurer–Cartan equation: $\mathrm{d}\alpha + \alpha \star \alpha = 0$.
- Gauge group action: $\lambda \cdot \alpha = \exp(\mathrm{ad}_\lambda)(\alpha) = \exp(\lambda) \star \alpha \star \exp(-\lambda)$.

Definition (Deformation gauge group)

Group-like elements: $\mathfrak{G} := (1 + \mathfrak{g}_0 , \star , 1)$.

Baker–Campbell–Hausdorff formula

$$(\mathfrak{g}_0\,,\log(\exp\star\exp)\,,0)\ \underset{\log}{\overset{\exp}{\underset{\longleftarrow}{\overset{\longrightarrow}{\cong}}}}\ \mathfrak{G}=(1+\mathfrak{g}_0\,,\star\,,1)$$

Theorem (Baker–Campbell–Hausdorff, 1902-1906)

$$\begin{aligned}\mathrm{BCH}(x,y)&:=\log\left(\exp(x).\exp(y)\right)\\&=x+y+\tfrac{1}{2}[x,y]+\tfrac{1}{12}[x,[x,y]]+\tfrac{1}{12}[y,[x,y]]+\cdots\\&\in\widehat{\mathrm{Lie}}(x,y)\subset\widehat{\mathrm{Ass}}(x,y)\ .\end{aligned}$$

Definition (Gauge group)

$\mathfrak{g}$ complete Lie algebra: $G:=(\mathfrak{g}_0\,,\mathrm{BCH}\,,0)$ topological group.

$$\mathrm{BCH}(x,y)\overset{\text{(Dynkin,1947)}}{=}\sum_{n\geqslant1}\frac{(-1)^{n-1}}{n}\sum_{\substack{p_i+q_i\geqslant1\\ i=\{1,\ldots,n-1\}}}\left(\frac{\mathrm{ad}_x^{p_1}\circ\mathrm{ad}_y^{q_1}\circ\cdots\circ\mathrm{ad}_x^{p_{n-1}}\circ\mathrm{ad}_y^{q_{n-1}}(x)}{(1+\sum_{i=1}^{n-1}p_i+q_i)p_1!q_1!\ldots p_{n-1}!q_{n-1}!}+\right.$$
$$\left.\sum_{p_n\geqslant0}\frac{\mathrm{ad}_x^{p_1}\circ\mathrm{ad}_y^{q_1}\circ\cdots\circ\mathrm{ad}_x^{p_{n-1}}\circ\mathrm{ad}_y^{q_{n-1}}\circ\mathrm{ad}_x^{p_n}(y)}{(p_n+1+\sum_{i=1}^{n-1}p_i+q_i)p_1!q_1!\ldots p_{n-1}!q_{n-1}!p_n!}\right)\ .$$

Deformation theory of $\mathcal{P}$-algebras

→ Endomorphism operad: $\mathrm{End}_A := (\{\mathrm{Hom}\,(A^{\otimes n}, A)\}, \{\circ_i\})$.

Definition ($\mathcal{P}$-algebra structure)

A representation of $\mathcal{P}$: a morphism of dg operads $\mathcal{P} \to \mathrm{End}_A$.

DEFORMATION THEORY OF $\mathcal{P}$-ALGEBRAS:

→ Koszul resolution: $\mathcal{P}_\infty := \Omega\mathcal{P}^{¡} \xrightarrow{\sim} \mathcal{P}$, with Koszul dual cooperad $\mathcal{P}^{¡}$.

$$\boxed{\{\mathcal{P}_\infty\text{-alg. on } A\} \cong \mathrm{Hom}_{\text{dg op.}}(\mathcal{P}_\infty, \mathrm{End}_A) \cong \mathrm{MC}\left(\underbrace{\mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{¡}, \mathrm{End}_A)}_{\text{convolution operad}}\right)}$$

⇒ Deformation theory of $\mathcal{P}$-alg.controlled by a complete dg pre-Lie alg.

$$\boxed{\text{associative alg.} \subset \text{pre-Lie alg.} \xrightarrow{-} \text{Lie alg.}}$$

Pre-Lie exponential/logarithm maps

$$\begin{tikzcd} G = (\mathfrak{g}_0\,,\mathrm{BCH}\,,0) \arrow[r, shift left=2, "?"] \arrow[r, phantom, "\cong"] & \mathfrak{G} = (1+\mathfrak{g}_0\,, ?\,, 1) \arrow[l, shift left=2, "?"] \end{tikzcd}$$

Definition (Pre-Lie exponential/logarithm maps)

- $\exp_\star(\lambda) := \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{(\cdots((\lambda\star\lambda)\star\lambda)\cdots)\star\lambda}_{n \text{ times}}$
- $\log_\star(1+\lambda) := \lambda - \frac{1}{2}\lambda\star\lambda + \frac{1}{4}\lambda\star(\lambda\star\lambda) + \frac{1}{12}(\lambda\star\lambda)\star\lambda + \cdots\,.$

Definition (Circle product)

$$(1+x)\circledcirc(1+y) := 1 + \sum_{n=0}^{\infty}\frac{1}{n!}\{x;\underbrace{y,\ldots,y}_{n \text{ times}}\} \quad \text{associative}$$

symmetric braces:

$$\begin{aligned}
\{x;\} &:= x \\
\{x;y_1\} &:= x\star y_1 \\
\{x;y_1,y_2\} &:= \{\{x;y_1\};y_2\}-\{x;\{y_1;y_2\}\}=(x\star y_1)\star y_2-x\star(y_1\star y_2) \\
\{x;y_1,\ldots,y_n\} &:= \{\{x;y_1,\ldots,y_{n-1}\};y_n\}-\sum_{i=1}^{n-1}\{x;y_1,\ldots,y_{i-1},\{y_i;y_n\},y_{i+1},\ldots,y_{n-1}\}.
\end{aligned}$$

Integration of pre-Lie algebras

Proposition (Dotsenko–Shadrin–V., 2016)

Complete dg pre-Lie algebra $(\mathfrak{g}, \star, \mathrm{d})$*:*

- $G = (\mathfrak{g}_0\,, \mathrm{BCH}\,, 0) \underset{\log_\star}{\overset{\exp_\star}{\rightleftarrows}} \mathfrak{G} := (1 + \mathfrak{g}_0\,, \circledcirc\,, 1)$ with $\cong$
- *Action of the deformation gauge group* $\mathfrak{G}$ *on* $\mathrm{MC}(\mathfrak{g})$*:*

$$(1+\lambda)\cdot\alpha = \big((1+\lambda)\star\alpha\big) \circledcirc (1+\lambda)^{-1} - \mathrm{d}\lambda \circledcirc (1+\lambda)^{-1}\ .$$

→ DELIGNE GROUPOID:

- Objects: $\mathcal{P}_\infty$-algebras,
- Morphisms: ∞-morphisms with 1^{st} component = id .

Applications

Theorem (Campos–Petersen–Robert-Nicoud–Wierstra, 2024)

- *The universal enveloping algebra functor* $\mathfrak{U}$*:*
$$\begin{array}{ccc} \textit{nilpotent Lie algebras} & \to & \textit{associative algebras} \\ \mathfrak{g} & \mapsto & \mathfrak{U}(\mathfrak{g}) := \mathrm{T}(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y]) \end{array}$$
detects isomorphisms.
- *The singular cochains algebra* $\left(C^{\bullet}_{\text{sing}}(X, \mathbb{Q}), \cup, \mathrm{d}\right)$ *encodes faithfully the rational homotopy type of* X*.*

Theorem (Dotsenko–Shadrin–Vaintrob–V., 2024)

- *Notion of quantum* CohFT_∞*.*
- *Universal symmetry group.*
- *contains Grothendieck–Teichmüller* GRT_1 *and Givental group.*

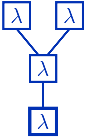
Limits of “general algebra”

- DEFORMATION GAUGE GROUP ACTION:

$$((1+\lambda)\star\alpha)\circledcirc(1+\lambda)^{-1}-\mathrm{d}\lambda\circledcirc(1+\lambda)^{-1}$$

- PARADIGM SHIFT:
  free pre-Lie algebra given by rooted trees [Chapoton–Livernet, 2001]

$$(1-\lambda)^{-1}=\sum_{\tau\in\mathrm{RT}}\frac{1}{|\mathrm{Aut}\,\tau|}\,\tau(\lambda)\ ,\quad \text{where } \tau(\lambda)=$$



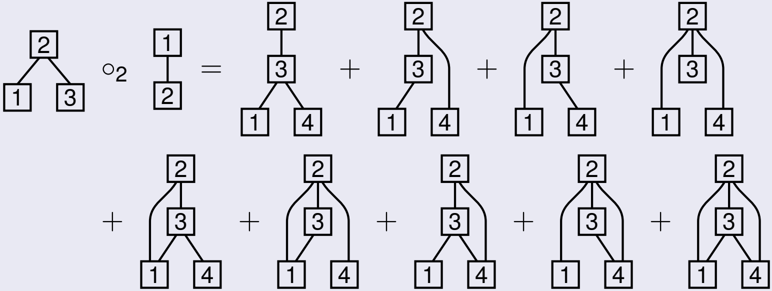
- $\mathcal{P}$ properad $\Rightarrow$ $\left(\prod_{n,m\in\mathbb{N}}\mathcal{P}(m,n),\star=\sum_{i,j}\circ_i^j\right)$ Lie-admissible algebra: $[a,b]=a\star b-b\star a$ satisfies the Jacobi identity.

  $\rightarrow$ Its iterations do not recover "all" the operations.

Lie-graph algebras

Definition (Operad Lie-graph)

top-to-bottom directed simple graphs (dsGra) with compositions:



→ different from the Kontsevich graph operad: creates edges.

Deformation theory of $\mathcal{P}$-bialgebras

Proposition (Campos-V., 2025)

- *$\mathcal{P}$ properad* $\Rightarrow \left(\prod_{n,m\in\mathbb{N}} \mathcal{P}(m,n), \{\star_\gamma\}_{\gamma\in\mathrm{dsGra}}\right)$ *is a Lie-graph algebra s.t.* $\star_{\begin{smallmatrix}\boxed{1}\\ | \\ \boxed{2}\end{smallmatrix}} = \star$ *is the Lie-admissible product.*
- *The operad* $\mathrm{Lie\text{-}graph}$ *is not finitely generated.*

DEFORMATION THEORY OF $\mathcal{P}$-BIALGEBRAS: representations $\mathcal{P} \to \mathrm{End}_A$ encoded by the complete dg Lie-graph algebra $\mathfrak{g}_{\mathcal{P},A} = \mathrm{Hom}_{\mathbb{S}}\left(\mathcal{P}^{¡}, \mathrm{End}_A\right)$,

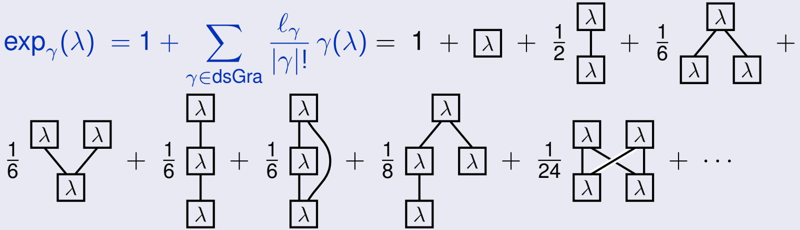
Maurer–Cartan equation $\mathrm{d}\alpha + \alpha \star \alpha = 0$.

$$\boxed{\text{associative alg.} \subset \text{pre-Lie alg.} \subset \text{Lie-graph alg.} \xrightarrow{-} \text{Lie alg.}}$$

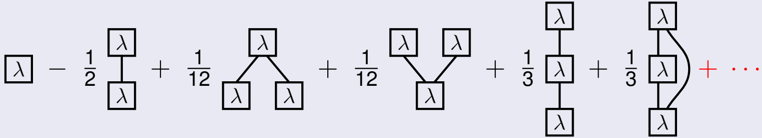
Lie-graph exponential/logarithm maps

Definition (Lie-graph exponential/logarithm maps)

- $\exp_\gamma(\lambda) = 1 + \sum_{\gamma \in \mathrm{dsGra}} \frac{\ell_\gamma}{|\gamma|!}\gamma(\lambda) =$



- $\log_\gamma(1+\lambda) =$



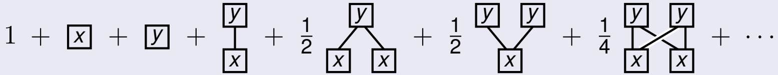
Integration of Lie-graph algebras

Proposition (Campos-V., 2025)

Complete dg Lie-graph algebra $(\mathfrak{g}, \{\star_\gamma\}_{\gamma\in\text{dsGra}}, \mathrm{d})$*:*

- $G = (\mathfrak{g}_0\,, \mathrm{BCH}\,, 0) \overset{\exp_\gamma}{\underset{\log_\gamma}{\overset{\longrightarrow}{\underset{\longleftarrow}{\cong}}}} \mathfrak{G} := (1+\mathfrak{g}_0\,, \circledcirc\,, 1)$, *where*

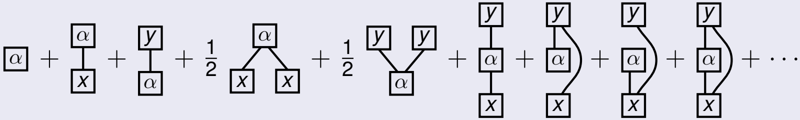
  $$(1+x)\circledcirc(1+y) = 1 + \sum_{\gamma\in\text{2-level dsGra}} \frac{1}{|\mathrm{Aut}(\gamma)|}\gamma(x,y) =$$

- *Action of the deformation gauge group* $\mathfrak{G}$ *on* $\mathrm{MC}(\mathfrak{g})$*:*

  $(1+\lambda)\cdot\alpha = (1+\lambda)\overset{\alpha}{\bowtie}(1+\lambda)^{-1} - (1+\lambda;\mathrm{d}\lambda)\circledcirc(1+\lambda)^{-1}$, *where*

  $$(1+x)\overset{\alpha}{\bowtie}(1+y) = \sum_{\gamma\in\bowtie\text{-dsGra}} \frac{1}{|\mathrm{Aut}(\gamma)|}\gamma(x,\alpha,y) =$$

Applications

→ DELIGNE GROUPOID:

- Objects: $\mathcal{P}_\infty$-bigebras,
- Morphisms: ∞-morphisms with 1st component = id .

Deformation gauge group $\left(1+\mathrm{Hom}_{\mathbb{S}}\left(\mathcal{P}^{¡}, \mathrm{End}_A\right)_0, \circledcirc, 1\right)$: any characteristic.

Theorem (Emprin, 2024)

- *complete formality classes for dg $\mathcal{P}$-bialgebras (after Kaledin)*

$$(A,\alpha) \xrightarrow{\sim} \bullet \xleftarrow{\sim} \bullet \cdots\cdots \bullet \xleftarrow{\sim} \bullet \xrightarrow{\sim} (H(A),\bar{\alpha})$$

- *descent property, "purity implies formality" (automorphism lift), etc.*

Theorem (Emprin–Takeda, 2025)

- *Intrinsic rational (co)formality of spheres, i.e. $C_*(\Omega S^n, \mathbb{Q})$, as pre-Calabi–Yau (bi)algebras with vanishing copairing: includes Poincaré duality.*
- *Not true in characteristic 2.*

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L_∞-algebras

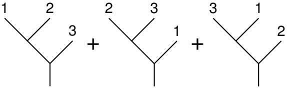
- DEFORMATION THEORY OF ∞-MORPHISMS OF $\mathcal{P}_\infty$-(BI)ALGEBRAS: encoded by L_∞-algebras, i.e. "weak Lie algebras".

Definition (L_∞-algebra)

$(\mathfrak{g}, \mathrm{d}, \{\ell_m\}_{m\geqslant 2})$: skew-symmetric operations $\ell_m \colon A^{\wedge m} \to A$, $|\ell_m| = m-2$, s.t.

$$\partial(\ell_m) = \mathrm{d}\circ \ell_m - (-1)^m \ell_m \circ \mathrm{d}_{A^{\wedge m}} = \sum_{\substack{p+q=m \\ 2\leqslant p,q \leqslant m}} \pm \sum_{\sigma \in \mathrm{Sh}_{p,q}^{-1}} (\ell_{p+1} \circ_1 \ell_q)^\sigma \ .$$

IN ARITY 3: $\partial \ell_3 = \mathrm{Jacobi}(\ell_2) =$

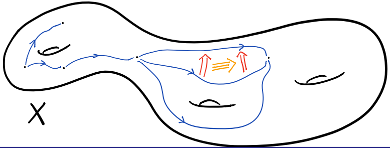


- MAURER–CARTAN EQUATION: $\mathrm{d}\alpha + \sum_{m\geqslant 2} \frac{1}{m!} \ell_m(\alpha, \ldots, \alpha) = 0$.
- GAUGE EQUIVALENCE:
  gauges: $\lambda \in \mathfrak{g}_0 \mapsto$ vector fields $\alpha \mapsto \sum_{m\geqslant 1} \frac{1}{(m-1)!} \ell_m(\alpha, \cdots, \alpha, \lambda) \in \mathrm{T}_\alpha \mathrm{MC}(\mathfrak{g})$.

∞-groupoids

gauge equivalence (tree-wise formula) $\overset{?}{\Leftarrow}$ "group up to homotopy" action

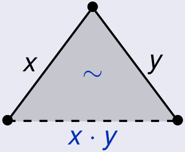
HEURISTIC: ∞-groupoid $\leftrightarrow$ topological space $\leftrightarrow$ Kan complex



Definition (∞-groupoid)

A *Kan complex*, i.e. a simplicial set $X_\bullet$ s.t.

$$\left\{ \begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook, "\wr"'] & X_\bullet \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \right\} \neq \emptyset$$



Integration of L_∞-algebras

complete L_∞-algebras $\xrightarrow{?}$ ∞-groupoids

Definition (Sullivan algebra)

Simplicial dg commutative algebra of polynomial differential forms on $|\Delta^n|$: $\Omega_\bullet = \{\Omega^*(\Delta^n)\}_{n\in\mathbb{N}}$

Theorem (Hinich, 1997)

$\mathrm{MC}_\bullet(\mathfrak{g}) := \mathrm{MC}\left(\mathfrak{g}\,\widehat{\otimes}\,\Omega_\bullet\right)$ *∞-groupoid s.t.* $\mathrm{MC}_0(\mathfrak{g}) \cong \mathrm{MC}(\mathfrak{g})$.

PROBLEM: $\mathrm{MC}_1(\mathfrak{g}) \not\supseteq$ gauges

SOLUTION: consider the simplicial Dupont contraction

$$\begin{tikzcd} \Omega^*(\Delta^\bullet) \arrow[loop left, "h_\bullet"] \arrow[r, shift left, "p_\bullet"] & C^*(\Delta^\bullet) \arrow[l, shift left, "i_\bullet"] \end{tikzcd}$$

Theorem (Getzler, 2009)

$\gamma_\bullet(\mathfrak{g}) := \mathrm{MC}_\bullet(\mathfrak{g}) \cap \ker h_\bullet \sim \mathrm{MC}_\bullet(\mathfrak{g})$ *∞-groupoid s.t.* $\gamma_1(\mathfrak{g}) =$ *gauges*

Effective integration of L_∞-algebras

ISSUE: not explicit ...

IDEA: transfer the simplicial commutative algebra structure from $\Omega^*(\Delta^\bullet)$ up to homotopy on $\mathrm{C}^*(\Delta^\bullet)$ and consider its linear dual (finite dim.).

Definition (Universal Maurer–Cartan L_∞-algebra)

The cosimplicial complete L_∞-algebra: $\mathfrak{mc}^\bullet := \left(\widehat{\mathrm{L}_\infty}(\mathrm{C}_*(\Delta^\bullet)), d\right)$.

Definition (Integration functor)

$$\mathrm{L} = \mathrm{Lan}_{\mathrm{Y}}\mathfrak{mc}^\bullet \ : \ \begin{tikzcd} \mathrm{sSet} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{L}_\infty\text{-alg.} \arrow[l, shift left] \end{tikzcd} \ : \ \mathrm{R}(\mathfrak{g}) := \mathrm{Hom}_{\mathrm{L}_\infty\text{-alg}}(\mathfrak{mc}^\bullet, \mathfrak{g})$$

Theorem (Robert-Nicoud–V., '20)

$$\gamma(\mathfrak{g}) \cong \mathrm{R}(\mathfrak{g}) \qquad \left\{ \begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook, "\wr"'] & \mathrm{R}(\mathfrak{g}) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \right\} \cong \mathfrak{g}_n \ni 0$$

$\Rightarrow$ algebraic ∞-groupoid [Nikolaus, 2011]: property → structure

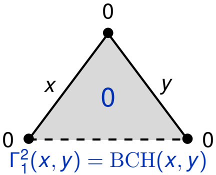
Higher Baker–Campbell–Hausdorff products

Definition (Higher Baker–Campbell–Hausdorff products)

The value at the missing $(n-1)$-simplex of the evaluation of the top dimensional cell of a horn in $\mathrm{R}(\mathfrak{g})$ by 0:

$$\mathrm{Hom}_{\mathsf{sSet}}\left(\Lambda^n_k, \mathrm{R}(\mathfrak{g})\right) \longrightarrow \mathfrak{g}_{n-1}\ , \quad x \longmapsto \Gamma^n_k(x)$$

EXAMPLE [Bandiera, 2014]:



$0 \in \mathrm{MC}(\mathfrak{g})$, $x, y \in \mathfrak{g}_0$.

Proposition (Robert-Nicoud–V., 2020)

$$\Gamma^n_k(x) = \sum_{\substack{\tau \in \mathrm{PaPRT} \\ \chi \in \mathrm{Lab}^{[n],k}(\tau)}} \prod_{\substack{\beta \text{ block of } \tau \\ \lambda^{\beta(\chi)}_{[n]} \neq 0}} \frac{(-1)^k}{\lambda^{\beta(\chi)}_{[n]} [\beta]!}\, \ell_\tau \left(x_{\chi(1)}, \ldots, x_{\chi(p)}; \sum_{l \neq k} (-1)^{k+l+1} x_{\widehat{l}} \right)$$

Applications

$\Rightarrow$ homotopy invariance (R(quasi-isomorphism)$=$homotopy equivalence), Berglund's Hurewicz theorem ($\pi_n(\mathrm{R}(\mathfrak{g}), \alpha) \cong H_n(\mathfrak{g}^\alpha)$), etc.

Theorem (Robert-Nicoud–V., 2020)

$X_\bullet$ pointed connected finite type simplicial set:
$\mathrm{RL}(X_\bullet)$ homotopy equivalent to Bousfield–Kan $\mathbb{Q}$-completion of $X_\bullet$

$\rightarrow$ [Buijs–Felix–Murillo–Tanré, 2020] Lie alg. case: L_∞-alg. are simpler.

$$\boxed{\text{Lie algebras} \subset \mathrm{L}_\infty\text{-algebras} \subset \text{absolute } \mathrm{EL}_\infty\text{-algebras}}$$

$\rightarrow$ "Koszul dual" to E_∞-alg., point-set model for spectral partition Lie alg.

Theorem (Roca i Lucio, 2024)

$X_\bullet$ a pointed connected finite type simplicial set
$\widetilde{\mathrm{R}}\widetilde{\mathrm{L}}(X_\bullet)$ homotopy equivalent to Bousfield–Kan $\mathbb{F}_p$-completion of $X_\bullet$

Poetry and mathematics ...

“Là, tout n’est qu’ordre et beauté,
Luxe, calme et volupté.”

Charles Baudelaire, *L’invitation au voyage*.

“What binds us to space-time is our rest mass, which prevents us from flying at the speed of light, when time stops and space loses meaning. In a world of light there are neither points nor moments of time; beings woven from light would live “nowhere” and “nowhen”; only poetry and mathematics are capable of speaking meaningfully about such things.”

Yuri. I. Manin, *Mathematics as metaphor*.

THANK YOU FOR YOUR ATTENTION!